Chapter 27: Lattices

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Overview

- Lattices used to analyze Bell-LaPadula, Biba constructions
- Consists of a set and a relation
- Relation must partially order set
  - Partial ordering < orders some, but not all, elements of set

Sets and Relations

- S set, R: $S \times S$ relation
  - If $a, b \in S$, and $(a, b) \in R$, write aRb
- Example
  - $I = \{ 1, 2, 3\}$; R is $\leq$
  - $R = \{ (1, 1), (1, 2), (1, 3), (2, 2), (2, 3), (3, 3) \}$
  - So we write $1 \leq 2$ and $3 \leq 3$ but not $3 \leq 2$

Relation Properties

- Reflexive
  - For all $a \in S$, aRa
  - On I, $\leq$ is reflexive as $1 \leq 1$, $2 \leq 2$, $3 \leq 3$
- Antisymmetric
  - For all $a, b \in S$, $aRb \wedge bRa \Rightarrow a = b$
  - On I, $\leq$ is antisymmetric
- Transitive
  - For all $a, b, c \in S$, $aRb \wedge bRc \Rightarrow aRc$
  - On I, $\leq$ is transitive as $1 \leq 2$ and $2 \leq 3$ means $1 \leq 3$

Bigger Example

- C set of complex numbers
- $a \in C \Rightarrow a = a_R + a_I i$, a_R, a_I integers
- $a \leq_C b$ if, and only if, $a_R \leq b_R$ and $a_I \leq b_I$
- $a \leq_C b$ is reflexive, antisymmetric, transitive
  - As $\leq$ is over integers, and a_R , a_I are integers

Partial Ordering

- Relation R orders some members of set S
  - If all ordered, it's total ordering
- Example
  - $\leq$ on integers is total ordering
  - $\leq_C$ is partial ordering on C (because neither $3+5i \leq_C 4+2i$ nor $4+2i \leq_C 3+5i$ holds)

Upper Bounds

- For a, $b \in S$, if u in S with aRu, bRu exists, then u is upper bound
  - Least upper if there is no $t \in S$ such that aRt, bRt, and tRu
- Example
  - For $1 + 5i$, $2 + 4i \in C$, upper bounds include $2 + 5i$, $3 + 8i$, and $9 + 100i$
  - Least upper bound of those is $2 + 5i$

Lower Bounds

- For a, $b \in S$, if l in S with lRa, lRb exists, then l is lower bound
  - Greatest lower if there is no $t \in S$ such that tRa, tRb, and lRt
- Example
  - For $1 + 5i$, $2 + 4i \in C$, lower bounds include 0, $-1 + 2i$, $1 + 1i$, and $1+4i$
  - Greatest lower bound of those is $1 + 4i$

Lattices

- Set S, relation R
  - R is reflexive, antisymmetric, transitive on elements of S
  - For every $s, t \in S$, there exists a greatest lower bound under R
  - For every $s, t \in S$, there exists a least upper bound under R

Example

- $S = \{ 0, 1, 2 \}$; $R = \leq$ is a lattice
  - R is clearly reflexive, antisymmetric, transitive on elements of S
  - Least upper bound of any two elements of S is the greater
  - Greatest lower bound of any two elements of S is the lesser

Picture

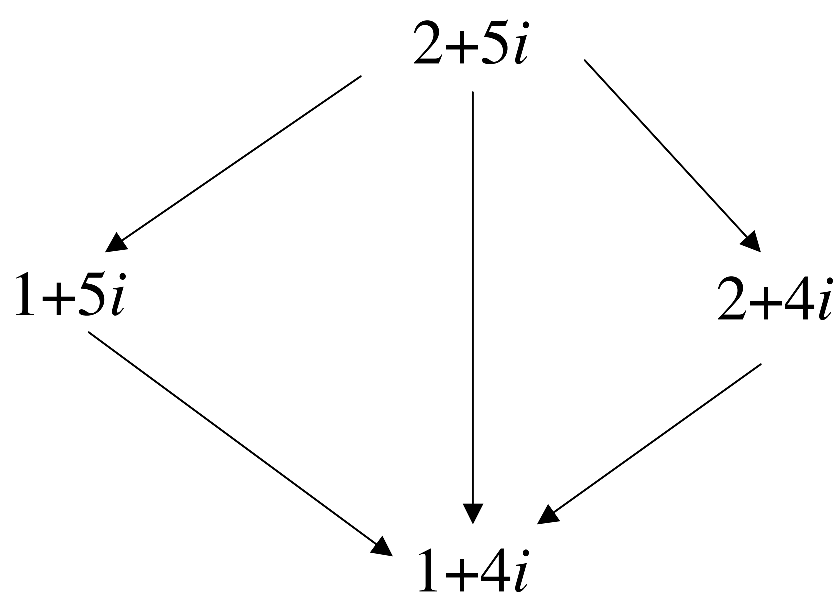
$$\begin{tikzcd} 2 \arrow[d] \\ 1 \arrow[d] \\ 0 \end{tikzcd}$$

Arrows represent ≤; total ordering

Example

- C, $\leq_C$ form a lattice
  - $\leq_C$ is reflexive, antisymmetric, and transitive
    - Shown earlier
  - Least upper bound for a and b:
    - $c_R = \max(a_R, b_R)$, $c_I = \max(a_I, b_I)$; then $c = c_R + c_I i$
  - Greatest lower bound for a and b:
    - $c_R = \min(a_R, b_R)$, $c_I = \min(a_I, b_I)$; then $c = c_R + c_I i$

Picture



Arrows represent $\leq_C$